Quantum Clifford-Hopf algebras for even dimensions

1994 J. Phys. A: Math. Gen. 27 845

(http://iopscience.iop.org/0305-4470/27/3/025)

J. Phys. A: Math. Gen. 27 (1994) 845–854. Printed in the UK

Quantum Clifford–Hopf algebras for even dimensions

E López

Instituto de Matemáticas y Física Fundamental, CSIC, Serrano 123, E–28006 Madrid, Spain

Received 28 June 1993, in final form 8 November 1993

Abstract. In this paper we study the quantum Clifford–Hopf algebras $\widehat{CH_q(D)}$ for even dimensions, D, and obtain their intertwiner R-matrices, which are elliptic solutions to the Yang–Baxter equation. In the trigonometric limit of these new algebras we find the possibility of connecting with extended supersymmetry. We also analyse the corresponding spin-chain Hamiltonian, which leads to Suzuki's generalized XY model.

1. Introduction

The quantum group structure plays an important role in the study of two-dimensional integrable models because R-matrices intertwining between diferent irreps of a quantum group provide solutions to the Yang–Baxter equation. Two important families of integrable models are the 6-vertex and 8-vertex solutions to the Yang–Baxter equation [1]. Whereas the 6-vertex solutions are intertwiners R-matrices for $U_q(\widehat{sl(2)})$, a quantum group interpretation for the elliptic 8-vertex family is not yet known.

Nevertheless, the 8-vertex regime is well understood for the particular class of solutions to the Yang–Baxter equation satisfying the free-fermion condition [2]

$$R^{00}_{00}(u)R^{11}_{11}(u) + R^{10}_{01}(u)R^{01}_{10}(u) = R^{11}_{00}(u)R^{11}_{00}(u) + R^{01}_{01}(u)R^{10}_{10}(u)\,. \tag{1}$$

Indeed, a quantum group-like structure has been found recently for the most general free fermionic elliptic 8-vertex model in a magnetic field. The matrix of its Boltzmann weights [3, 4] acts as intertwiner for the affinization of a quantum Hopf deformation of the Clifford algebra in two dimensions, noted $\widehat{CH_q(2)}$ [5].

A major interest of the free fermionic solutions to the Yang–Baxter equation is in their connection, in the 6-vertex limit ($R^{11}_{00}(u) = R^{00}_{11} = 0$), with $N = 2$ supersymmetric integrable models. The free fermionic 6-vertex solutions are given by the R-matrix intertwiners between nilpotent irreps of the Hopf algebra $U_\epsilon(\widehat{sl(2)})$, with $\epsilon^4 = 1$ (the nilpotent irreps are a special case of the cyclic representations that enlarge the representation theory of $U_\epsilon(\widehat{sl(2)})$ when ϵ is a root of unity). In the trigonometric limit the R-matrix for $\widehat{CH_q(2)}$ becomes that for $U_\epsilon(\widehat{sl(2)})$, $\epsilon^4 = 1$.

In this article we construct the quantum Clifford–Hopf algebras $\widehat{CH_q(D)}$ for even dimensions $D \geqslant 2$, generalizing the results in [5]. This general case is interesting because it yields one of the rare examples of elliptic R-matrices. The R-matrices we find admit several spectral parameters, due to the structure of $\widehat{CH_q(D)}$ as a Drinfeld twist [6] of the tensor product of several copies of $\widehat{CH_q(2)}$. The possibility of connecting with extended supersymmetry in the trigonometric limit of $\widehat{CH_q(D)}$, and a related supersymmetric integrable model are analysed in section 3. Finally, in section 4, we study the spin-chain Hamiltonian associated with these algebras. The model obtained represents several XY Heisenberg chains in an external magnetic field [7] coupled among them in a simple way. Though the coupling is simple it can be a starting point to get a quantum group structure for more complicated models built through the coupling of two XY or XX models (Bariev model [8], one-dimensional Hubbard model). The last part of this section is devoted to showing the equivalence of this model—under some restrictions—with a generalized XY model proposed by Suzuki in relation to the two-dimensional dimer problem [9].

2. The quantum Clifford algebra

A Clifford algebra $C(\eta)$ related to a quadratic form or metric η is the associative algebra generated by the elements $\{\Gamma_\mu\}_{\mu=1}^{D}$, which satisfy

$$\{\Gamma_\mu, \Gamma_\nu\} = 2\eta_{\mu\nu}\mathbf{1} \qquad \mu, \nu = 1, \ldots, D. \tag{2}$$

The quantum Clifford–Hopf algebra $CH_q(D)$ [5] is a generalization and quantum deformation of $C(\eta)$, generated by elements Γ_μ, Γ_{D+1} (the analogue of γ_5 for the Dirac matrices) and new central elements E_μ ($\mu = 1, \ldots, D$) verifying

$$\begin{aligned} &\Gamma_\mu^2 = \frac{q^{E_\mu} - q^{-E_\mu}}{q - q^{-1}} \qquad \Gamma_{D+1}^2 = \mathbf{1} \\ &\{\Gamma_\mu, \Gamma_\nu\} = 0 \quad \mu \neq \nu \qquad \{\Gamma_\mu, \Gamma_{D+1}\} = 0 \\ &[E_\mu, \Gamma_\nu] = [E_\mu, \Gamma_{D+1}] = [E_\mu, E_\nu] = 0 \qquad \forall \mu, \nu . \end{aligned} \tag{3}$$

The charges E_μ result from elevating the components of the metric η from numbers to operators. The generator Γ_{D+1} will play a similar role to $(-1)^F$, where F is the fermion number operator. Although for the standard Clifford algebra D represents the dimension of the spacetime, in our case D is only a parameter labelling (3). The algebra $CH_q(D)$ is a Hopf algebra with the following co-multiplication Δ, antipode S and co-unit ϵ:

$$\begin{aligned} &\Delta(E_\mu) = E_\mu \otimes \mathbf{1} + \mathbf{1} \otimes E_\mu && S(E_\mu) = -E_\mu && \epsilon(E_\mu) = 0 \\ &\Delta(\Gamma_\mu) = q^{E_\mu/2}\Gamma_{D+1} \otimes \Gamma_\mu + \Gamma_\mu \otimes q^{-E_\mu/2} && S(\Gamma_\mu) = \Gamma_\mu \Gamma_{D+1} && \epsilon(\Gamma_\mu) = 0 \\ &\Delta(\Gamma_{D+1}) = \Gamma_{D+1} \otimes \Gamma_{D+1} && S(\Gamma_{D+1}) = \Gamma_{D+1} && \epsilon(\Gamma_{D+1}) = 1 . \end{aligned} \tag{4}$$

The irreducible representations of $CH_q(D)$ are in one-to-one correspondence with those of the Clifford algebra $C(\eta)$ for all possible signatures of the metric η, in D (D even) or $D+1$ (D odd) dimensions, respectively. They are labelled by complex parameters $\{\lambda_\mu\}_{\mu=1}^{D}$, the eigenvalues of the Casimir operators $K_\mu = q^{E_\mu}$. From now on we restrict ourselves to the case D even, $D = 2M$.

The irreps of $CH_q(2M)$ are isomorphic to the tensor product of M $CH_q(2)$ irreps, being their dimension 2^M. Thus, a basis for $CH_q(2M)$ can be obtained from the $CH_q(2)^{\otimes M}$ generators as follows (γ_α, $E_\alpha (\alpha = 1, 2)$, $\gamma_3 \in CH_q(2)$):

$$\begin{aligned} &\Gamma_{2(n-1)+\alpha} = \gamma_3 \otimes \cdots \otimes \gamma_3 \otimes \overset{n)}{\gamma_\alpha} \otimes 1 \otimes \cdots \otimes 1 \qquad n = 1, \ldots, M \quad \alpha = 1, 2 \\ &E_{2(n-1)+\alpha} = 1 \otimes \cdots \otimes 1 \otimes E_\alpha \otimes 1 \otimes \cdots \otimes 1 \qquad \Gamma_{D+1} = \gamma_3 \otimes \cdots \otimes \gamma_3 . \end{aligned} \tag{5}$$

The Hopf algebra $CH_q(2M)$ is related to the tensor product $CH_q(2)^{\otimes M}$ by a Drinfeld twist B [6]

$$\Delta_{CH_q(2M)}(g) = B\Delta_{CH_q(2)^{\otimes M}}(a)B^{-1} \qquad \forall g \in CH_q(2M) \tag{6}$$

where the operator $B \in CH_q(2)^{\otimes M} \otimes CH_q(2)^{\otimes M}$ acting on the tensor product of two $CH_q(2M)$ irreps is defined by

$$B = (-1)^{F*F} \qquad F*F = \sum_{1\leqslant j<i\leqslant M} (1\otimes\cdots\otimes \overset{i)}{f} \otimes\cdots\otimes 1)\otimes(1\otimes\cdots\otimes \overset{j)}{f}\otimes\cdots\otimes 1) \tag{7}$$

where $f = 0$ (boson), 1 (fermion) is the fermion number for the two vectors in a $CH_q(2)$ irrep. The reason for introducing the operator B in formula (6) is that the co-multiplication in $CH_q(2)^{\otimes M}$ treats each factor $CH_q(2)$ separately. This can be represented by a twist between the $CH_q(2)$ pieces of a $CH_q(2M)$ irrep. Since one of the vectors in a $CH_q(2)$ irrep behaves as a fermion, this twist has the effect of introducing some signs that we represent by the operator B (figure 1).

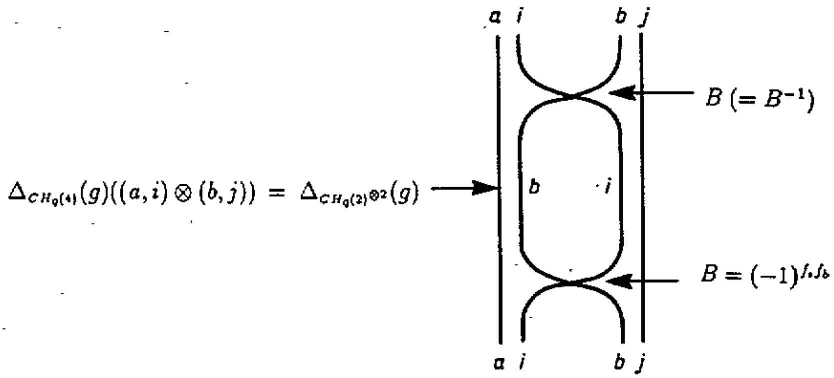

Figure 1. Graphical representation of the expression (6) for $CH_q(4)$; (a, i) denote the vectors in a $CH_q(2)^{\otimes 2}$ irrep, index a corresponding to the first $CH_q(2)$ and i to the second.

Next we introduce a sort of affinization of the Hopf algebra $CH_q(D)$. The generators of this new algebra $\widehat{CH_q(D)}$ are $\Gamma_\mu^{(i)}$, $E_\mu^{(i)}$ $(i = 0, 1)$ and Γ_{D+1}, verifying (3) and (4) for each value of i. We also impose that the anticommutator $\{\Gamma_\mu^{(1)}, \Gamma_\nu^{(2)}\}$ belong to the centre of $\widehat{CH_q(D)}$ $\forall\mu, \nu$.

We now give the explicit realization of $\widehat{CH_q(2)}$. It is a useful example, and it will provide us with the building blocks for any D. A two-dimensional irrep π_ξ of $\widehat{CH_q(2)}$ is labelled by $\xi = (z, \lambda_1, \lambda_2) \in C^3$ and reads as follows:

$$\pi_\xi(\gamma_1^{(0)}) = \left(\frac{\lambda_1^{-1}-\lambda_1}{q-q^{-1}}\right)^{1/2}\begin{pmatrix}0 & z^{-1}\\ z & 0\end{pmatrix} \qquad \pi_\xi(\gamma_1^{(1)}) = \left(\frac{\lambda_1-\lambda_1^{-1}}{q-q^{-1}}\right)^{1/2}\begin{pmatrix}0 & z\\ z^{-1} & 0\end{pmatrix}$$

$$\pi_\xi(\gamma_2^{(0)}) = \left(\frac{\lambda_2^{-1}-\lambda_2}{q-q^{-1}}\right)^{1/2}\begin{pmatrix}0 & -\mathrm{i}z^{-1}\\ \mathrm{i}z & 0\end{pmatrix} \qquad \pi_\xi(\gamma_2^{(1)}) = \left(\frac{\lambda_2-\lambda_2^{-1}}{q-q^{-1}}\right)^{1/2}\begin{pmatrix}0 & -\mathrm{i}z\\ \mathrm{i}z^{-1} & 0\end{pmatrix} \tag{8}$$

$$\pi_\xi(\gamma_3) = \begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}$$

$$\pi_\xi(q^{E_1^{(0)}}) = \lambda_1^{-1} \qquad \pi_\xi(q^{E_1^{(1)}}) = \lambda_1$$

$$\pi_\xi(q^{E_2^{(0)}}) = \lambda_2^{-1} \qquad \pi_\xi(q^{E_2^{(1)}}) = \lambda_2\,.$$

For the affine $C\widehat{H_q(2M)}$ we can define a straightforward generalization of (5). It allows us to introduce M different affinization parameters $\{z_n\}_{n=1}^M$, one for each $C\widehat{H_q(2)}$ piece:

$$\Gamma^{(i)}_{2(n-1)+\alpha} = \gamma_3\otimes\cdots\otimes\gamma_3\otimes\gamma_\alpha^{(i)}\otimes 1\otimes\cdots\otimes 1 \qquad n=1,\ldots,M \quad \alpha=1,2 \quad i=0,1$$

$$E^{(i)}_{2(n-1)+\alpha} = 1\otimes\cdots\otimes 1\otimes E_\alpha^{(i)}\otimes 1\otimes\cdots\otimes 1 \tag{9}$$

$$\Gamma_{D+1} = \gamma_3\otimes\cdots\otimes\gamma_3\,.$$

The intertwiner R-matrix for two irreps with labels $\xi = \{z_n, \lambda_{2n-1}, \lambda_{2n}\}_{n=1}^M$ is defined by the condition

$$R_{\xi_1\xi_2}\Delta_{\xi_1\xi_2}(g) = \Delta_{\xi_2\xi_1}(g)R_{\xi_1\xi_2} \qquad \forall g\in C\widehat{H_q(2M)} \tag{10}$$

with $\Delta_{\xi_1\xi_2} = \pi_{\xi_1}\otimes\pi_{\xi_2}(\Delta)$. Since (6) remains true for any element $g\in C\widehat{H_q(2M)}$, the intertwiner R-matrix between two irreps (which furthermore satisfies the Yang–Baxter equation) is given by [6]

$$R_{CH_q(2M)}(u_1,\ldots,u_M) = B R_{CH_q(2)^{\otimes M}}(u_1,\ldots,u_M)B^{-1}$$
$$R_{CH_q(2)^{\otimes M}}(u_1,\ldots,u_M) = R^{(1)}_{CH_q(2)}(u_1)\ldots R^{(M)}_{CH_q(2)}(u_M)\,. \tag{11}$$

The matrices $R^{(n)}_{CH_q(2)} = R_{\xi_1^{(n)}\xi_2^{(n)}}$ ($\xi^{(n)} = (z_n,\lambda_{2n-1},\lambda_{2n})$) are the $C\widehat{H_q(2)}$ intertwiners

$$R_{00}^{00} = 1-e(u_n)e_1e_2 \qquad R_{11}^{11} = e(u_n)-e_1e_2$$

$$R_{01}^{10} = e_1-e(u_n)e_2 \qquad R_{10}^{01} = e_2-e(u_n)e_1$$

$$R_{01}^{01} = R_{10}^{10} = (e_1\mathrm{sn}_1)^{1/2}(e_2\mathrm{sn}_2)^{1/2}(1-e(u_n))/\mathrm{sn}(u_n/2) \tag{12}$$

$$R_{00}^{11} = R_{11}^{00} = -\mathrm{i}k(e_1\mathrm{sn}_1)^{1/2}(e_2\mathrm{sn}_2)^{1/2}(1+e(u_n))\mathrm{sn}(u_n/2)$$

where $e(u_n) = \mathrm{cn}(u_n)+\mathrm{i}\,\mathrm{sn}(u_n)$ is the elliptic exponential of modulus k_n, $e_i = e(\psi_i^n)$, $\mathrm{sn}_i = \mathrm{sn}(\psi_i^n)$ ($i=1,2$) and u_n, ψ_i^n are elliptic angles depending on the labels $\xi_i^{(n)}$ (see [5] for details).

There is a constraint on the irrep labels so that (12) is indeed their intertwiner

$$\frac{2(\lambda_{2n-1}-\lambda_{2n})}{(1-\lambda_{2n-1}^2)^{1/2}(1-\lambda_{2n}^2)^{1/2}(z_n^2-z_n^{-2})}=k_n \qquad n=1,\ldots,M\,. \tag{13}$$

All the $R^{(n)}_{CH_q(2)}$ matrices are independent and commute among them. It's remarkable that the spectral curve (13) of irreps that admit an intertwiner is parametrized by M independent elliptic moduli k_n. Indeed, some of them can be in the elliptic regime and others in the trigonometric ($k=0$). The matrix $R_{CH_q(2M)}$ can be thought of as the scattering matrix for objects composed of M different kinds of particles. There is real interaction when two equal particles scatter from each other, given by $R^{(n)}_{CH_q(2)}$; otherwise there is only a sign coming from their statistics and represented by the operator B (figure 2).

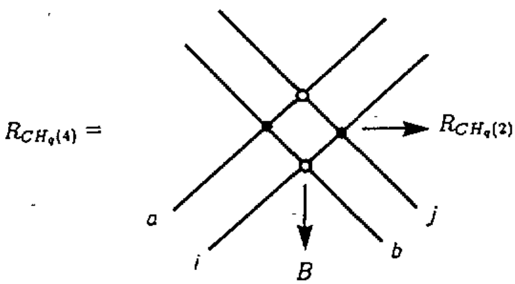

Figure 2. Graphical representation of the $CH_q(4)$ R-matrix.

Finally, note that the R-matrix (12) coincides with the Boltzmann weights for the most general 8-vertex free-fermionic solution to the Yang–Baxter equation in non-zero magnetic field [3, 4].

3. Extended supersymmetry

In order to analyse the connection of $\widehat{CH_q(2M)}$ with supersymmetry algebras, we will study the limit in which the R-matrix (12) becomes trigonometric. Let us consider first the case $D=2$ in detail. This case turns out to be related to an $N=2$ (two supersymmetry charges) integrable Ginzburg–Landau model. We shall also give a heuristic motivation for the construction of the Hopf algebra $\widehat{CH_q(2)}$ based on its trigonometric 6-vertex limit.

The 6-vertex free fermionic solutions are given by the intertwiner R-matrix between nilpotent irreps of $U_\epsilon(\widehat{sl(2)})$, $\epsilon^4=1$ ($\Rightarrow \epsilon=i$) [10]. In a $U_{\epsilon=i}(sl(2))$ nilpotent irrep the values of the special Casimirs are $Q_\pm^2=0$ ($Q_\pm=S_\pm\epsilon^{\pm H/2}$) and $K^2=\lambda^2$ arbitrary ($K=\epsilon^H$); namely, they are the highest-weight cases of the cyclic irreps. Furthermore when $\epsilon^4=1$ the anticommutator $\{Q_+,Q_-\}$ also belongs to the centre, suggesting the connection with a Clifford algebra through the mixing of the positive and negative root generators $Q_\pm$. The total fermion number is conserved in the 6-vertex solutions to the Yang–Baxter equation, but it is not in the elliptic regime. Hence a non-trivial mixing is needed to represent the elliptic regime. The Hopf algebra $CH_q(2)$ assigns different *central elements* $[E_1]_q$, $[E_2]_q$ to the square of the generators γ_1, γ_2 respectively, in such a way that the mixing can only be undone (trigonometric limit) when $E_1 = E_2 = E$. It implies $k = 0$ in (13). For the affine $\widehat{CH_q(2)}$ this limit leads to $U_{\epsilon=i}(\widehat{sl(2)})$ (this statement is only rigurous for the affine case): i.e. $R_{CH_q(2)}$ becomes the R-matrix intertwiner for $U_{\epsilon=i}(\widehat{sl(2)})$, provided the labels of the two algebras are related by $\lambda = q^E$.

Using the generators $Q_\pm, \overline{Q}_\pm \in U_{\epsilon=i}(\widehat{sl(2)})$, we can define an $N = 2$ supersymmetry algebra with topological extension $T_\pm$ [11, 12]

$$\begin{aligned} &Q_\pm^2 = \overline{Q}_\pm^2 = \{Q_\pm, \overline{Q}_\pm\} = 0 \\ &\{Q_\pm, \overline{Q}_\mp\} = 2T_\pm \qquad |T_\pm| = [E]_q \\ &\{Q_+, Q_-\} = 2mz^2 \qquad \{\overline{Q}_+, \overline{Q}_-\} = 2m\,z^{-2} \end{aligned} \tag{14}$$

satisfying the Bogomolnyi bound $|T_\pm| = m$. The free fermionic condition (1) ensures the $N = 2$ invariance of the R-matrix. Moreover, the $N = 2$ part of the scattering matrix for the solitons of the Ginzburg–Landau superpotential $W = X^{n+1}/(n+1) - \beta X$ [13] is given by R-matrices of $U_{\hat{q}}(\widehat{gl(1,1)})$ with $\hat{q}^{2n} = 1$ [14], or equivalently by those of $U_{\epsilon=i}(\widehat{sl(2)})$ between nilpotent irreps with labels $\lambda = \hat{q}$ [15].

The Ginzburg–Landau models have a particular importance in the context of $N = 2$ supersymmetry, since they allow one to classify a wide variety of $N = 2$ superconformal field theories [16]. Of great interest are the relevant perturbations of these theories giving massive integrable models, as happens for the superpotential $W(X) = X^{n+1}/(n+1) - \beta X$. We would now like to make plausible in this context why the supersymmetry algebra (14) has a non-trivial co-multiplication. In a $N = 2$ Ginzburg–Landau model, the superpotential enters explicitly in the SUSY commutators through

$$\begin{aligned} &\{Q_+, \overline{Q}_+\} = \Delta W \qquad \{Q_-, \overline{Q}_-\} = \Delta W^* \\ &\Delta W = W(X^j) - W(X^i) \end{aligned} \tag{15}$$

with $X(-\infty) = X^i$, $X(\infty) = X^j$ and X^i, X^j minima of W. Let's call $K_{(i,i+l)}$ the soliton going from X^i to X^j, where $l = j - i$. It is straightforward to see that ΔW depends on both l and i. Naively, the dependence on i was not expected since all the solitons with the same l are equivalent. For the superpotential proposed it is possible to obtain a supersymmetric algebra without this dependence, at the price of re-absorbing it in a non-trivial quantum group comultiplication

$$\begin{aligned} &\Delta(Q_\pm) = q^{\pm E}\gamma_3 \otimes Q_\pm + Q_\pm \otimes \mathbf{1} \\ &\Delta(\overline{Q}_\pm) = q^{\mp E}\gamma_3 \otimes \overline{Q}_\pm + \overline{Q}_\pm \otimes \mathbf{1}\,. \end{aligned} \tag{16}$$

On the other hand, it is worth noting the relation of (16) with the fermion number of the solitons. In the solitonic sectors, the fermion number operator acquires a fractional constant piece due to the interaction of the fermionic degrees of freedom with the solitonic background. The fractional piece of the fermion number in a soliton sector $K_{(i,j)}$, is given by [17, 18]

$$f = -\frac{1}{2\pi}(\mathrm{Im}\ln W''(X))\,|_{X^i}^{X^j} = \frac{s}{n} \qquad s = 1, \ldots, n-1\,. \tag{17}$$

The relation with $CH_q(2)$ labels is $q^E = e^{i\pi s/n}$. Therefore $q^{\pm E}\gamma_3$ in (16) would be the analogue of $e^{\pm i\pi F}$, where F is the fermion number operator. This interpretation fails for $\Delta(\overline{Q}_\pm)$, where the signs are interchanged, leading, in fact, to a quantum group structure instead of a Lie superalgebra.

Let us return to building extended supersymmetry algebras from the general $\widehat{CH_q(2M)}$, in the same sense as above. The trigonometric limit of $\widehat{CH_q(2M)}$ is obtained as an independent trigonometric limit in each $\widehat{CH_q(2)}$ piece. Then the affine Hopf algebra $\widehat{CH_q(2M)}$ becomes, in essence, the anticommuting tensor product of M $U_{\epsilon=i}(\widehat{sl(2)})$ factors, each with its own spectral parameter. Imposing that the eigenvalues of all the central charges E_i and the spectral parameters z_i $(i = 1, \ldots, M)$ coincide, we get M copies of the same structure (14), $\{Q_\pm^{(i)}, \overline{Q}_\pm^{(i)}, T_\pm^{(i)} = T_\pm\}_{i=1}^M$. Therefore we find an $N = 2M$ supersymmetry algebra with M topological charges. Indeed, the dimension of a $\widehat{CH_q(2)}$ irrep is 2^M, as is needed to saturate the Bogomolnyi bound $|T_\pm^{(i)}| = |T_\pm| = m$.

Besides, we have seen that the $\widehat{CH_q(2M)}$ irreps can be thought of as collections of M independent solitons $\widehat{CH_q(2)}$. Let us consider the more general trigonometric limit with equal values of the central charges E_i, but arbitrary spectral parameters z_i $(i = 1, \ldots, M)$. Then the charges

$$Q_\pm^T = \sum_{i=1}^{M} Q_\pm^{(i)} \qquad \overline{Q}_\pm^T = \sum_{i=1}^{M} \overline{Q}_\pm^{(i)} \tag{18}$$

verify the commutation relations of $N = 2$ supersymmetry (14). In fact, (14) is satisfied even if we allow different central charges E_i. However, in this case the comultiplication doesn't preserve the expression (18) of $Q_\pm^T, \overline{Q}_\pm^T$.

4. Generalized XY spin chains

The quantum group structure plays an important role in two-dimensional statistical models, since R-matrix intertwiners provide systematic solutions to the integrability condition, the Yang–Baxter equation. In this way integrable models can be built associated with a quantum group, allowing one to connect integrability with an underlying symmetry principle. As noted above, the intertwiner R-matrix for the Clifford–Hopf algebra $\widehat{CH_q(2)}$ reproduces the 8-vertex free-fermion model in magnetic field. In this section we will analyse the model defined by the algebras $\widehat{CH_q(D)}$ for general $D = 2M$. Following the transfer-matrix method, the study of a two-dimensional statistical model is equivalent to that of its corresponding spin chain. The L-site Hamiltonian for a periodic chain defined by the $\widehat{CH_q(2M)}$ Hopf algebras is given by (provided that $R(0) = \mathbf{1}$)

$$\begin{aligned} H &= \sum_{j=1}^{L} i\frac{\partial}{\partial u} R_{j,j+1}(u)\,|_{u=0} \\ H &= \sum_{j=1}^{L}\sum_{n=1}^{M}\{(J_x^n\sigma_{x,j}^n\sigma_{x,j+1}^n + J_y^n\sigma_{y,j}^n\sigma_{y,j+1}^n)\sigma_{z,j}^{n+1}\ldots\sigma_{z,j}^M\sigma_{z,j+1}^1\ldots\sigma_{z,j+1}^{n-1} + h^n\sigma_{z,j}^n\} \end{aligned} \tag{19}$$

where σ_a^n ($a = x, y, z$ $n = 1, \ldots, M$) are M sets of Pauli matrices, and the constants J_x^n, J_y^n, h^n depend on the quantum labels of the irreps whose intertwiner is R

$$\begin{aligned} J_x^n &= 1 + \Gamma^n \qquad J_y^n = 1 - \Gamma^n \qquad n = 1, \ldots, M \\ \Gamma^n &= k_n \mathrm{sn}(\psi^n) \qquad h^n = 2\mathrm{cn}(\psi^n)\,. \end{aligned} \tag{20}$$

The requirement $R(0) = \mathbf{1}$ implies $\psi_1^n = \psi_2^n = \psi^n$.

The Hamiltonian (19) can be diagonalized through a Jordan–Wigner transformation and its excitations behave as free fermions (massless when $J_x^n = J_y^n$, massive otherwise). This model provides M groups of Pauli matrices $\sigma_{a,j}^n$ ($a = x, y, z$) for each site j on the chain, so it behaves as having M layers with an XY model defined in each layer. The factors $(\sigma_{z,j}^{k+1} \ldots \sigma_{z,j}^M \sigma_{z,j+1}^1 \ldots \sigma_{z,j+1}^{k-1})$ make the fermionic excitations on different layers anticommute. Thus the algebra $C\widehat{H_q(2M)}$ provides a way to put different non-interacting fermions in a chain with a quantum group interpretation.

When $M = 1$, H reduces to the Hamiltonian of an XY Heisenberg chain in an external magnetic field h, that is the spin chain associated with the 8-vertex free-fermion model [7]:

$$H = \sum_{j=1}^{L} \{J_x \sigma_{x,j}\sigma_{x,j+1} + J_y \sigma_{y,j}\sigma_{y,j+1} + h\sigma_{z,j}\}\,. \tag{21}$$

The aim of this section is to show that the above model is equivalent under some restrictions to the generalized integrable XY chain proposed and solved in [9],

$$\widetilde{H} = -\sum_{k=1}^{K}\sum_{j=1}^{L'} (\tilde{J}_x^k \sigma_{x,j}\sigma_{x,j+k} + \tilde{J}_y^k \sigma_{y,j}\sigma_{y,j+k})\sigma_{z,j+1} \ldots \sigma_{z,j+k-1} + h\sum_{j=1}^{L'} \sigma_{z,j} \tag{22}$$

finding in this way a quantum group structure for this integrable model. The Hamiltonian (22) can also be diagonalized with a Jordan–Wigner transformation and its quasi-particles behave as free fermions. The main application of the generalized XY model is the problem of covering a surface with horizontal and vertical dimers. Indeed, the ground state of $\widetilde{H}$ for a particular choice of parameters reproduces the two-dimensional pure dimer problem [9], first solved in terms of a Pfaffian [19].

To see the relation between H and $\widetilde{H}$, let us choose identical XY models on each layer of the former chain

$$J_x^n = J_x \qquad J_y^n = J_y \qquad h^n = h \quad n = 1, \ldots, M \tag{23}$$

and rearrange the spin labels to form a single-layer chain

$$\sigma_{a,j}^n = \sigma_{a,j+n} \qquad n = 1, \ldots, M \quad a = x, y, z\,. \tag{24}$$

Then the Hamiltonians H and $\widetilde{H}$ coincide if in the latter we set

$$\tilde{J}_x^k = -J_x \delta_{M,k} \qquad \tilde{J}_y^k = -J_y \delta_{M,k} \qquad k = 1, \ldots, K\,. \tag{25}$$

The general $\widetilde{H}$ (22) is obtained by adding Hamiltonians $H^{(M)}$ derived from $C\widehat{H_q(2M)}$ R-matrices. The fact that this sum is also solvable relies on setting equal parameters in each $H^{(M)}$ (this is the same condition that leads to $N = 2M$ supersymmetry in the trigonometric limit of $C\widehat{H_q(2M)}$). Therefore, the affine quantum Clifford–Hopf algebras $C\widehat{H_q(2M)}$ encode the hidden quantum group for the generalized XY spin chain (22).

5. Comments

We have studied the quantum Clifford algebras $\widehat{CH_q(2M)}$ in connection with extended supersymmetry and with statistical integrable models.

It is worth noting that the Hamiltonian derived from $\widehat{CH_q(4)}$ in the trigonometric regime and without magnetic field is the limiting case $U \to \infty$ of the two-layer chain [8]:

$$H = -\tfrac{1}{2}\sum_{j=1}^{L}\{(\sigma_x^j\sigma_x^{j+1} + \sigma_y^j\sigma_y^{j+1})(1 - U\tau_z^{j+1}) + (\tau_x^j\tau_x^{j+1} + \tau_y^j\tau_y^{j+1})(1 - U\sigma_z^j)\}. \qquad (26)$$

The coupling between the two layers in this model implies real interaction, so the excitations are not free fermions, and the ground state presents spontaneous magnetization (if $U \neq 0, \infty$). It can still be solved by Bethe ansatz techniques, but an R-matrix interpretation for it is not known. The algebra $\widehat{CH_q(4)}$ gives us a simple way of coupling two XY models. Perhaps it would be possible to twist (may be in a way related to a quantum deformation proposed recently for the Clifford algebras [20]) and break the full set of generators to a shorter set giving a quantum group structure for this model.

We have built extended supersymmetric algebras from the $\widehat{CH_q(2M)}$ generators in the trigonometric limit. The Clifford–Hopf algebras can be thought of as elliptic generalizations of supersymmetry (the anticommutators of charges that give the momentum P and $\overline{P}$ get deformed in the elliptic case, but are still central elements). It would be interesting to analyse what deformation of the Poincaré group one gets in such a way.

Acknowledgments

The author would like to thank A Berkovich, R Cuerno, C Gómez and G Sierra for discussions and encouragement. This work is supported by MEC fellowship AP91 34090983.

References

[1] Baxter R J 1982 *Exactly Solved Models In Statistical Mechanics* (New York: Academic)
[2] Fan C and Wu F Y 1970 *Phys. Rev.* B 2 723
[3] Felderhof B U 1973 *Physica* **66** 279
[4] Bazhanov V V and Stroganov Yu G 1985 *Teor. Mat. Fiz.* **62** 337
[5] Cuerno R, Gómez C, López E and Sierra G 1993 *Phys. Lett.* **307B** 56
[6] Drinfeld V G 1989 Quasi-Hopf algebras and Knizhnik–Zamolodchikov equations *Preprint* ITP-89-43B
[7] Lieb E H, Schultz T D and Mattis D C 1961 *Ann. Phys., NY* **16** 407
[8] Bariev R Z 1991 *J. Phys. A: Math. Gen.* **24** L549
[9] Suzuki M 1971 *Prog. Theor. Phys.* **46** 1337; 1971 *Phys. Lett.* **34A** 338
[10] Berkovich A, Gómez C and Sierra G 1993 Spin-anisotropy commensurable chains: Quantum group symmetries and $N = 2$ SUSY *Preprint* IMAFF-93-1; 1994 *Nucl. Phys.* B to appear
[11] Bernard D and LeClair A 1990 *Phys. Lett.* **247B** 309
[12] LeClair A and Vafa C 1992 Quantum affine symmetry as generalized supersymmetry *Preprint* HUTP-92/A045
[13] Fendley P and Intriligator K 1992 *Nucl. Phys.* B 380 265
[14] Rozansky L and Saleur H 1992 *Nucl. Phys.* B **376** 461
Kauffman L H and Saleur H 1991 *Commun. Math. Phys.* **141** 293
[15] Murakami J 1992 *Int. J. Mod. Phys.* A **7** Suppl. 1B 765
[16] Vafa C ann Warner N P 1989 *Phys. Lett.* **218B** 51

[17] Fendley P and Intriligator K 1992 *Nucl. Phys.* B **372** 533
[18] Goldstone J and Wilczek F 1981 *Phys. Rev. Lett.* **47** 986
Niemi A J and Semenoff G W 1986 *Phys. Rep.* **135** 99
[19] Kasteleyn P W 1962 *Physica* **27** 1209; 1963 *J. Math. Phys.* **4** 287
[20] Brzezinsky T, Papaloucas L C and Rembielinski J 1993 Quantum clifford algebras *Preprint* HEP-TH/9306061